Adiabatic approach to mean-first-passage-time computation in bistable potential with colored noise

T. G. Venkatesh and L. M. Patnaik
Microprocessor Applications Laboratory, Indian Institute of Science, Bangalore 560 012, India
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An approach to compute the mean-first-passage time (MFPT) in bistable systems driven by colored noise is presented. The approach is valid in the limit of large but finite noise correlation time and finite noise strength, a case for which no satisfactory theory exists at present. Our approach is a modification of the fluctuating-potential theory proposed in Phys. Rev. Lett. **61**, 7 (1988). Excellent agreement is found between our results for MFPT and that of the simulation results of Mannella, Palleschi, and Grigolini [Phys. Rev. A **42**, 5946 (1990)]. Interesting similarity between stochastic resonance and the colored-noise problem is brought out.

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I. INTRODUCTION

Bistable systems driven by colored noise (BSDCN) acts as a model for a wide variety of physical, chemical, natural, and engineering systems and has drawn wide attention in recent years [1]. BSDCN is one of the simplest problems, coming under the broad class of problems called "nonlinear non-Markovian stochastic processes." The equation governing BSDCN is

$$\dot{x}=x-x^3+\xi(t)\ , \tag{1}$$

where $\xi(t)$ is a mean zero Ornstein-Uhlenbeck (OU) noise with correlation

$$\langle \xi(t)\xi(t')\rangle=(D/\tau)\exp(-|t-t'|/\tau)\ ,$$

D is the noise strength, and τ is the noise correlation time. The dynamics of Eq. (1) can be visualized as an overdamped motion of a ball in a bistable potential $V(x)=x^4/4-x^2/2$, in the presence of the OU noise. The bistable potential has an unstable state at $x=0$ and two stable states at $x=+1$, $x=-1$ separated by a barrier of height $\frac{1}{4}$. Two quantities of interest are (1) the stationary probability density function (SPDF) of x and (2) the mean first passage time (MFPT) for x to go from one stable state to another. Presence of OU noise in Eq. (1) makes x non-Markovian, and this coupled with the nonlinear nature of Eq. (1) makes it difficult to compute the statistical quantities, especially the SPDF of x and the MFPT. There is no simple Fokker-Planck-like equation for the evolution of the probability density function of x. Also, we do not have an exact, closed-form formula for the MFPT valid for a general colored noise. Various approximate theories for computing the MFPT, valid in the limit of small D and small or large τ, have been put forth [2]. We restrict our discussions to the large-τ limit. All existing theories valid in the limit of $(\tau/D)\rightarrow\infty$ predict the exponent of the MFPT to be $(2\tau)/(27D)$ [3–8] (but see also Ref. [9]). But the convergence of theories of Refs. [3–8] to the limit $(\tau/D)\rightarrow\infty$ is very slow [10(a)], and this limit is inaccessible by practical numerical simulation [10(b)]. Hence the validation of theories of Refs. [3–8] can only be done at large but finite τ. For finite τ, theories of Refs. [3–6] underestimate the MFPT, whereas Ref. [7], and especially Ref. [8], reproduce the MFPT fairly well [5,11,12]. However, Refs. [7] and [8], which are based on the path-integral formalism, invoke smallness of D for carrying out the steepest-descent approximation while computing the MFPT. Thus all existing theories fail in the limit of finite τ and finite D. We attempt to fill this gap through this paper.

The paper is organized as follows. As our approach is an adaptation of the fluctuating potential theory (FPT), we first review FPT and its modifications in Sec. II. The adiabatic approach to the MFPT computation is presented in Sec. III. Here we derive an analytical formula for the escape rate (the inverse of MFPT) in the limit of finite D and finite τ. In Sec. IV our results for the MFPT are compared with that of the simulation results reported in Ref. [12] and the outcomes are discussed. In Sec. V we reconsider the equation for the escape rate derived in Sec. III. In Sec. VI we bring out the interesting similarity between the stochastic resonance phenomenon and the colored-noise problem. Finally, our results are summarized in Sec. VII.

II. FLUCTUATING POTENTIAL THEORY AND ITS MODIFICATIONS

The FPT uses the fact that for large τ, noise is slower than the system variable x of Eq. (1) [3]. Invoking adiabatic elimination principle, one assumes that x will always be found in the instantaneous potential minima determined by $\xi(t)$. As $\xi(t)$ increases, the Brownian particle initially at $x=-1$ adiabatically follows the potential minima. When $\xi(t)$ reaches $\xi_c\equiv+2/(3\sqrt{3})$, the potential minima merges with the potential maxima at $x=-1/\sqrt{3}$, and transition to the positive well takes place immediately.

de la Rubia *et al.* have modified the FPT based on the following argument [4]. When $\xi(t)$ reaches ξ_c, potential at $x=-1/\sqrt{3}$ becomes flat and sufficient drift may not be available for the Brownian particle to roll down from $x=-1/\sqrt{3}$ to $x=0$ in a time of $O(\tau)$. This results in an unsuccessful transition for finite τ. Hence for a successful transition from the negative well to take place $\xi(t)$ should reach a larger, τ-dependent value, ξ_τ, determined by the equation

$$\int_{-1/\sqrt{3}}^{0} \frac{dx}{x-x^3+\xi_\tau}=\tau \, .$$

Reference [5] generalizes the idea of FPT and that of Ref. [4] and predicts the value of $\xi(t)$ needed for a transition from the negative well to be $\xi_c(1+(1/\tau))$. There is a definite improvement in predicting the MFPT by Refs. [4] and [5], but they still underestimate the MPFT for large τ [11].

Our own result [13] indicates that the value of $\xi(t)$ needed for a successful transition to be even larger than the value predicted by Refs. [4] and [5]. This fact will be used to arrive at the limitations of the theory presented in this paper. Our above-mentioned claim on the value of $\xi(t)$ needed for transition is based on two reasons. First, we point out that the FPT breaks down when $\tau|V''(x)|\le 1$. Since $V''(x)$ [the second derivative of the bistable potential, $V''(x)=3x^2-1$] vanishes at $x=\pm 1/\sqrt{3}$, x does not adiabatically follow $\xi(t)$ for $\tau\ge 1$ in the regions $-\sqrt{(\tau+1)/(3\tau)}\le x\le -\sqrt{(\tau-1)/(3\tau)}$ and $+\sqrt{(\tau-1)/(3\tau)}\le x\le +\sqrt{(\tau+1)/(3\tau)}$. The second point is that for an Ornstein-Uhlenbeck noise of correlation time τ typical realizations rise and fall exponentially in a time of $O(\tau)$. Using these two points it can be shown that for finite τ, x will not reach $-1/\sqrt{3}$ when $\xi(t)$ reaches ξ_c. Further, for finite τ, we have argued in Ref. [13] that it is better to view the escape mechanism as caused by noise spikes with a rise and fall time of $O(\tau)$. Simulation of the bistable system with such noise spikes show that typically $\xi(t)$ has to reach a value higher than what has been predicted by Refs. [4] and [5] for a transition to take place.

III. ADIABATIC APPROACH

We first recall the way by which an Ornstein-Uhlenbeck process of strength D and correlation time τ is generated from a white noise [1]. It is given by the equation

$$\dot{\xi}(t)=(-1/\tau)\xi(t)+\eta(t) \, , \tag{2}$$

where $\langle \eta(t)\rangle=0$ and $\langle \eta(t)\eta(t')\rangle=(2D/\tau^2)\delta(t-t')$.

Equation (2) suggests that in the limit $\tau\rightarrow\infty$, $\xi(t)$ can be viewed as a superposition of a smooth exponentially varying value $\overline{\xi(t)}$, varying over a time of $O(\tau)$, and a white noise of strength D/τ^2. The effect of this white noise is then to make x in Eq. (1) fluctuate around the instantaneous potential minima determined by $\overline{\xi(t)}$. In other words, in the limit $\tau\rightarrow\infty$, one can assume the effect of the smooth variation $\overline{\xi(t)}$ on x and the effect of white noise on x to be independent. However, this distinction becomes meaningless for small and intermediate τ, that is, for $\tau V''(x=0)\le 1$. When D is finite, the fluctuation of x caused by this white noise may lead to a transition to the other well even prior to $\overline{\xi(t)}$ reaching ξ_c. The white-noise-induced escape at a given $\overline{\xi(t)}$ occurs at a rate determined by D/τ^2 and the instantaneous barrier height. When $\overline{\xi(t)}$ is near 0, the white-noise-induced escape rate is small, but the escape rate from the negative (positive) well increases as $\overline{\xi(t)}$ increases towards ξ_c $(-\xi_c)$, due to the decrease in the barrier height. In fact, for finite τ, when the MFPT due to the white noise of strength D/τ^2 at a given $\overline{\xi(t)}$ becomes comparable with τ, we can assume the transition to take place in a time of $O(\tau)$.

Yet another way of looking at the effect of $\xi(t)$ on x at finite D and finite τ is as follows. The instantaneous potential of Eq. (1) is given by $V(x)=x^4/4-x^2/2-\xi(t)x$, where $\xi(t)$ has two parts: (1) a smooth exponentially varying part $\overline{\xi(t)}$ and (2) a white-noise part of strength D/τ^2. The effect of $\overline{\xi(t)}$ is to slowly tilt the potential [i.e., in a time of $O(\tau)$], and the effect of the white noise is to randomly vibrate the potential around the tilted position determined by $\overline{\xi(t)}$. Therefore even before $\overline{\xi(t)}$ reaches ξ_c, the random vibrations due to the white noise can cause the well in which the particle is present to disappear. But note that since variations in the white noise $\eta(t)$ are faster compared to that of $\overline{\xi(t)}$, $\eta(t)$ [and hence $\xi(t)$] has to reach larger values compared to $\overline{\xi(t)}$ so that x can roll down deterministically from its position fixed by $\overline{\xi(t)}$ to 0.

Further, we note that FPT is valid only in the limit $(\tau/D)\rightarrow\infty$ and $D\rightarrow 0$. It can be seen that all the modifications of FPT proposed so far [4,5,13] correct the FPT when τ is finite, but still demand $(\tau/D)\rightarrow\infty$ by letting $D\rightarrow 0$. For large but finite τ, only in the limit $D\rightarrow 0$ is it correct to assume that a transition will not occur unless $\overline{\xi(t)}$ reaches ξ_c. However, if both τ and D are finite, transitions can occur even prior to $\overline{\xi(t)}$ reaching ξ_c, and the below-mentioned adiabatic approach needs to be invoked. In this context the validity of the white noise and the small-τ theories can be recognized to be in the limit $(\tau/D)\rightarrow 0$ and with a nonvanishing D.

Henceforth, by a transition we always mean transition from the negative well to the positive well, unless otherwise specified. In the light of the above discussion, we then propose the mean escape rate of x, in the limit of large τ and finite D, to be given by the formula

$$\Re=\int_0^{\xi_c} d\xi \, P_s(\xi)R(\xi) \, , \tag{3}$$

where ξ_c $[=+2/(3\sqrt{3})]$ is the value of $\xi(t)$ when the negative well of the potential corresponding to Eq. (1) vanishes. $P_s(\xi)$ is the stationary probability density function of the OU process defined in Eq. (2),

$$P_s(\xi)=\frac{1}{\sqrt{2\pi D/\tau}}\exp\left[\frac{-\xi^2\tau}{2D}\right] . \tag{4}$$

$R(\xi)$ is the escape rate of x due to the white noise of strength D/τ^2 from the instantaneous potential minima to the instantaneous potential maxima, when $\overline{\xi(t)}=\xi$. Equation (3) arises due to the following reason. We want the escape rate of x at stationarity (i.e., stationarity of x as well as ξ). At any arbitrary time the probability that $\overline{\xi(t)}=\xi$ is $P_s(\xi)$. The escape rate of x is then the escape rate $R(\xi)$ due to white noise of strength D/τ^2, given $\overline{\xi(t)}=\xi$, averaged over the SPDF of $\overline{\xi(t)}$. One need not get confused, by viewing the MFPT of x to be the average of the MFPT of $\overline{\xi(t)}$ to ξ, added to the MFPT of x to the other well due to white noise at $\overline{\xi(t)}=\xi$. This will in turn demand that the MFPT due to white noise to the other well be shorter than the correlation time τ, leading to a conceptual problem. The usage of $P_s(\xi)$ and the escape rate instead of the MFPT avoids the possible confusion. Note that in Eq. (3) we have not integrated ξ over the ranges $\xi \geq \xi_c$ and $-\infty < \xi \leq 0$. We will analyze Eq. (3) by including these ranges of integration in Sec. V and rule out these ranges of integration based on physical grounds.

Changing the variable from ξ to x in Eq. (3), where x is the position of the instantaneous potential minima when $\overline{\xi(t)}=\xi$, we have

$$\Re = \int_{-1}^{-1/\sqrt{3}} dx \frac{1}{\sqrt{2\pi D/\tau}} \exp\left[\frac{-(x^3-x)^2\tau}{2D}\right] \times \frac{|3x^2-1|}{3} R(x) . \quad (5)$$

$|3x^2-1|$ is the Jacobian of transformation $\xi = x^3 - x$. Factor 3 arises because $x^3 - x - \xi = 0$ has three roots for x, in the range $0 \leq \xi < \xi_c$. $R(x)$ is the Kramers escape rate due to white noise of strength D/τ^2 when x is the potential minima and is given by

$$R(x) = \frac{\sqrt{2}}{\pi} \exp\left[\frac{-\tau^2 \Delta U(x)}{D}\right] , \quad (6)$$

where $\Delta U(x) = U(x_2) - U(x_3)$ is the barrier height when x is the potential minima, and $U(x) = (x^4/4) - (x^2/2) - (x^3 - x)x$.

The potential maxima x_2 and the potential minima x_3 $(=x)$ are two of the three roots of the equation $x^3 - x = \xi$, which are given by [14]

$$x_1 = 2\sqrt{\tfrac{1}{3}}\cos(\alpha/3) ,$$

$$x_{2,3} = -2\sqrt{\tfrac{1}{3}}\cos\left[\frac{\alpha}{3} \pm \frac{\pi}{3}\right] ,$$

where

$$\alpha = \cos^{-1}\left[\frac{(x^3-x)\sqrt{27}}{2}\right] .$$

We have performed the numerical integration of Eq. (5) for various values of τ and D and the results obtained are discussed in the next section. Before concluding this section we clarify an important point. The Kramers escape rate $R(x)$ given in Eq. (6) should actually read

$$R(x) = \frac{\sqrt{|V''(x_2)|V''(x_3)}}{\pi} \exp\left[\frac{-\tau^2 \Delta U(x)}{D}\right] . \quad (7)$$

However, as ξ approaches ξ_c both the potential minima and the potential maxima approach each other and $V''(x)$ at both these points approaches zero. In this limit the prefactor of $R(x)$ tends to zero and the MFPT explodes. As it is difficult to fix the ranges for the value of potential minima x beyond which Eq. (6) becomes invalid, we are forced to use the approximate equation (6) throughout the range $-1 \leq x \leq -1/\sqrt{3}$. Further, we will show at the end of Sec. V that the distribution of $P_s(\xi)R(\xi)$ has to be confined well within the range $0 \leq \xi \leq \xi_c$ for Eq. (3) to be valid. A consequent limitation on Eq. (5) restricts its integrant to be distributed well within $-1 \leq x \leq 1/\sqrt{3}$. As Eq. (6) for $R(x)$ becomes accurate with this restriction on Eq. (5), we are consistent in the usage of Eq. (6) in Eq. (5).

IV. COMPARISON OF MFPT OBTAINED BY ADIABATIC APPROACH WITH THE SIMULATION RESULTS

Figure 1 shows the comparison of the results for the MFPT, $\Re^{-1}$, obtained by numerical integration of Eq. (5) with the simulation results for T_{bot} of Mannella, Palleschi, and Grigolini [12]. T_{bot} is the MFPT from the bottom of one well to that of another well. Simulation results for the MFPT (T_{bot}) for $D=0.2$ and 0.3 are taken from Table I of Ref. [12], whereas for $D=0.25$, 0.35, and 0.4, T_{bot} is computed from Table III of Ref. [12] using the fitting parameters (see Ref. [12] for full details of this computation). $\Re^{-1}$ actually corresponds to T_{top}, the MFPT to the top of the barrier. Since $T_{\mathrm{top}} \simeq T_{\mathrm{bot}}$ in the ranges of D and τ considered, and due to the nonavailability of T_{top} for $D=0.25$, 0.35, and 0.4, we have compared $\Re^{-1}$ with T_{bot} of Ref. [12]. We find excellent agreement between our results with that of Ref. [12] for $D=0.3$ over the entire range $1 \leq \tau \leq 10$ considered. An agreement within 20% error is found for $D=0.25$ and 0.35. For $D=0.2$ and 0.4 the agreement is rather poor.

We offer the following explanation for the behavior of the results shown in Fig. 1. From Eq. (3) we see that the escape rate of x is the integral over the product of $P_s(\xi)$ and $R(\xi)$. For values of ξ close to $\xi=0$, $R(\xi)$ is less whereas $P_s(\xi)$ is more. This situation reverses near $\xi(t)=\xi_c$. Hence, for finite D the peak of the product $P_s(\xi)R(\xi)$ occurs at $0 \leq \xi \leq \xi_c$. Let us examine the ex-

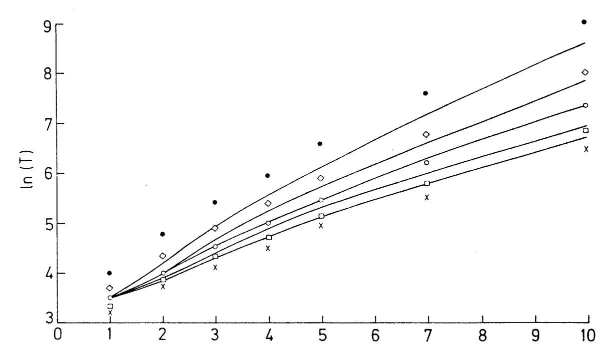

FIG. 1. ln(T) as a function of τ for various values of D. Solid lines are results of Eq. (5) with $(-1 \leq x \leq -1/\sqrt{3})$ as the range of integration of x. From top to bottom $D=0.2$, 0.25, 0.3, 0.35, and 0.4. Symbols are results of Ref. [12]: solid circles, $D=0.2$; diamonds, $D=0.25$; open circles, $D=0.3$; squares, $D=0.35$; and crosses, $D=0.4$.

ponential factor occurring within the integral of Eq. (5), i.e.,

$$\exp\left[\frac{-\tau}{D}\left[\frac{(x^3-x)^2}{2}+\tau\Delta U(x)\right]\right].$$

In the limit $(\tau/D)\rightarrow\infty$, the integral in Eq. (5) is dominated by the values of x which make $\{(x^3-x)^2/2+\tau\Delta U(x)\}$ smaller. As x increases from -1 to $(-1/\sqrt{3})$, $(x^3-x)^2$ increases, whereas $\tau\Delta U(x)$ decreases. For large τ, $\tau\Delta U(x)$ dominates $(x^3-x)^2$. Hence for large τ and in the limit $(\tau/D)\rightarrow\infty$ the integral in Eq. (5) is dominated by values of x close to $-1/\sqrt{3}$ and $\Re$ approaches the leading order, the well-known limit $\exp[(-2\tau)/(27D)]$ [3–8].

Dominance of the integral in Eq. (5) by values of x close to $-1/\sqrt{3}$ in the limit $(\tau/D)\rightarrow\infty$ introduces the following errors in Eq. (5).

(1) As we have already pointed out, the FPT breaks down for finite τ near $x=\mp 1/\sqrt{3}$. It can be shown that the Brownian particle lags behind the potential minima and will not reach $\mp 1/\sqrt{3}$ when $\xi(t)$ reaches $\pm 2/(3\sqrt{3})$ for a typical noise realization of finite correlation time. Hence $P_s(x)$, which gives the SPDF of the position of the potential minima when $\overline{\xi(t)}=\xi$, does not represent the SPDF of the position of the Brownian particle $P_s(x_{\mathrm{BP}})$, for finite τ in the region of breakdown of the FPT. $P_s(x_{\mathrm{BP}})$ is the one we should actually use, along with the escape rate from x_{BP} to the potential maxima in Eq. (5). In fact, $P_s(x_{\mathrm{BP}}=y)\ll P_s(x=y)$ for y in the region of breakdown of the FPT. By assigning higher probability for transitions over smaller barrier heights (i.e., for smaller MFPT's), Eq. (5) underestimates the correct MFPT in the limit of finite τ and $(\tau/D)\rightarrow\infty$.

(2) When escape occurs to the potential maxima for value of ξ less than but near ξ_c, the potential between $x=-1/\sqrt{3}$ and $x=0$ is flat, and transition to the other well may not be successful at finite τ (de la Rubia *et al.*'s argument [4]). This factor leads to further underestimation of the MFPT by Eq. (5) in the limit of finite τ and $(\tau/D)\rightarrow\infty$. But note that for finite D escape to potential maxima occurs at values of ξ considerably less than ξ_c and the deterministic drift in the region $(-1/\sqrt{3})\le x\le 0$ will be sufficient for the transition to the positive well.

The above-mentioned two points explain the reason for the underestimation of the MFPT by Eq. (5) (as shown in Fig. 1) in the limit of finite τ and $(\tau/D)\rightarrow\infty$ (or $D\rightarrow 0$ at finite τ). Hence Eq. (5) is unsuitable for finite τ and $D\rightarrow 0$, and this limit requires an entirely different treatment [13]. Note, however, that the limit $(\tau/D)\rightarrow\infty$, favoring transition over lower barrier heights, does not invalidate the usage of the Kramers formula for escape rate. This is because the strength of the white noise $(=D/\tau^2)$ also decreases in the limit $(\tau/D)\rightarrow\infty$. For very large D but finite τ, Eq. (5) is obviously not valid because Kramers formula for escape rate itself becomes invalid, even when the potential minima is near $x=-1$. This explains the disagreement of our results with that of Ref. [12] for $D=0.4$.

V. EQUATION FOR $\Re$ RECONSIDERED

We now reconsider Eq. (3) by extending the range of integration of ξ to $-\infty<\xi<+\infty$. In fact, to have mathematical consistency one should integrate the SPDF of ξ over the entire range of ξ. First we include the range $-\infty<\xi\le 0$ for integration in Eq. (3). Integration over this range amounts to the fact that we are accounting for the escape that takes place from the negative well even when it is deeper than the positive well. From a first look one may conclude that there is no reason why one should not include this contribution to the MFPT. But we rule out this range of integration on physical grounds. First we note that for $-\infty<\xi\le-\xi_c$, the positive well vanishes and escape to the positive well loses meaning. So we have to include only the range $-\xi_c\le\xi\le 0$ in Eq. (3). We have performed numerical integration of Eq. (5) corresponding to the limit $-\xi_c\le\xi\le\xi_c$ and found that the resulting MFPT explodes, instead of adding a small correction to our previous result shown in Fig. 1. This behavior is of no surprise and it is an expected result for the following reason. $P_s(\xi)$ is symmetrical about $\xi=0$, whereas $R(\xi)$ becomes very small for $\xi<0$. So integration of Eq. (3) in the range $-\xi_c\le\xi\le+\xi_c$ assigns the same probability to the escape rate $R(+\xi)$ and $R(-\xi)$. This accounts for the explosion of the MFPT. Both $+\xi$ and $-\xi$ are equally probable, but the escape rate at $+\xi$ is much higher than the escape rate at $-\xi$. Naturally the first passage of the Brownian particle to the barrier top will occur only when $\overline{\xi(t)}=+\xi$ rather than at $\overline{\xi(t)}=-\xi$. This explains the reason for the omission of the region $-\infty<\xi\le 0$ in Eq. (3).

We now consider the range $\xi_c\le\xi<\infty$, in which case the negative well vanishes and the barrier height becomes zero. However, the escape rate $R(x)$ cannot be fixed easily for $\overline{\xi(t)}\ge\xi_c$. For $D\rightarrow 0$, and for finite τ, $\overline{\xi(t)}$ has to typically reach a value $\hat{\xi}$ larger than ξ_c for a transition to take place [4,5,13], and in this limit the escape rate is zero for $\xi_c\le\xi\le\hat{\xi}$ and 1 for $\xi\ge\hat{\xi}$. However, for large D the escape rate is almost close to 1 for $\xi\ge\xi_c$. For finite D, in which we are presently interested, we then approximately fix the escape rate $R(\xi)$ as 0.5 for $\xi\ge\xi_c$.

We performed the numerical integration of Eq. (3) in the range $\xi\ge 0$ [correspondingly, $-1\le x\le -1/\sqrt{3}$ and $x\ge 2/\sqrt{3}$ in Eq. (5), with $R(x)=0.5$ for $x\ge 2/\sqrt{3}$]. The corresponding results for T_{bot} are again compared with that of Ref. [12] and is shown in Fig. 2. We find that the agreement between our results and those of Ref. [12] improves for $D=0.2$, and notably for $D=0.25$. However, the results for $D=0.3$ and 0.35 deviate more. Better agreement of the MFPT for smaller values of D is due to the fact that in the limit $D\rightarrow 0$ the contribution to the MFPT when $\xi\ge\xi_c$ is appreciable and we are accounting for this in the results shown in Fig. 2. By the same token, as we have argued that first passage will take place only when $\xi(t)$ is at $+\xi$ rather than at $-\xi$, it can be seen that for large D the first passage will take place from values of $\overline{\xi(t)}$ near the peak of the product $P_s(\xi)R(\xi)$ rather than for $\xi\ge\xi_c$. This factor is responsible for the discrepancies of results for $D=0.3$ and 0.35, as shown in Fig. 2.

We conclude that Eq. (3) is valid only when the majori-

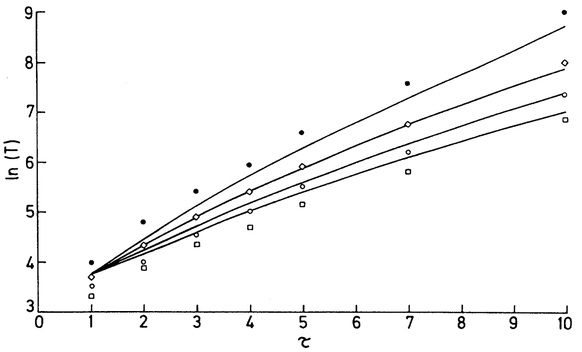

FIG. 2. $\ln(T)$ as a function of τ for various values of D. Solid lines are results of Eq. (5) with ($-1 \le x \le -1/\sqrt{3}$ and $x \ge 2/\sqrt{3}$) as the range of integration of x. From top to bottom $D=0.2$, 0.25, 0.3, and 0.35. Symbols are results of Ref. [12]: solid circles, $D=0.2$; diamonds, $D=0.25$; open circles, $D=0.3$; and squares, $D=0.35$.

ty of distribution of the product $P_s(\xi)R(\xi)$ lies within the range $0 \le \xi \le \xi_c$. Hence we exclude the range of integration $\xi \ge \xi_c$ in Eq. (3) and restrict the validity of Eq. (3) to finite D. Consequently, in the case of Eq. (5) we confine the range of integration of x to be $-1 \le x \le -1/\sqrt{3}$ and require its integrant to be distributed well within this range. We recall that with this restriction on Eq. (5), we are consistent in using Eq. (6) for $R(x)$.

VI. CONNECTION BETWEEN STOCHASTIC RESONANCE AND THE COLORED-NOISE PROBLEM

Note the interesting similarity between our approach and the adiabatic approach used in the context of stochastic resonance (SR) [15]. The validity of the adiabatic approach to SR requires the external forcing frequency to be smaller than the Kramers escape rate due to the white noise. Analogously, our approach requires smallness of $1/\tau$ to allow for slow variation of the potential and also simultaneously requires largeness of D/τ^2 to have a large escape rate caused by the white noise. These two conflicting requirements can only be satisfied for a narrow range of D and τ. This very clearly explains the behavior of our results shown in Fig. 1.

We further discuss the striking similarity between SR and a bistable system driven by strongly correlated noise (BSDSCN). We have seen that at large but finite τ OU noise can be viewed as a superposition of a slowly varying value $\overline{\xi(t)}$ and a white noise of strength (D/τ^2). Both SR and BSDSCN have the commonality of a bistable system and a white noise, the only difference being that in SR we have a periodic forcing whereas in BSDSCN the forcing is random. We encounter a resonance in the escape rate in SR for certain ranges of the parameters involved. It is then natural to ask whether a resonance like phenomenon occurs in the case of BSDSCN also.

We now present arguments based on physical grounds supporting the existence of a resonance phenomenon in the case of BSDSCN. For very large τ noise is mostly deterministic and the effect of the white noise (of strength D/τ^2) part of OU noise is negligible and escape occurs only when $\overline{\xi(t)}$ reaches $\pm\xi_c$. For $\tau \to 0$, OU noise approaches white noise and the $\overline{\xi(t)}$ is always 0. Escape occurs at Kramers rate due to the white noise part of OU noise with the barrier height $U(x=0)-U(x=-1)=\frac{1}{4}$, corresponding to $\overline{\xi(t)}=0$. In the intermediate τ regime escape occurs due to both the white noise factor as well as due to $\overline{\xi(t)}$ factor. Hence we can expect a value of τ when both these factors are in resonance.

To put it differently we can identify two adiabatic limits, one at small τ (where x is slower than noise) and another at large τ (where noise is slower than x, except in the region of invalidity of FPT). In each adiabatic limit, one of two factors of the OU noise—the white noise or $\overline{\xi(t)}$— plays the dominant role and the other a minor role in causing transitions. At the point of separation of these adiabatic regimes both these factors may aid each other, enhancing the escape phenomenon. Note that the value of τ separating the two adiabatic limits is space dependent and hence occurs at different τ for different x.

A theoretical analysis for predicting a resonance at intermediate τ is not as straightforward as in the case of SR. Resonance occurs in SR due to the synchronization in the enhanced net escape rate (from the shallower well to the deeper well averaged over one-half of the period of the forcing signal) with the external frequency. In the case of colored noise there is no periodicity to maintain synchronization. However, statistically speaking we can assume the OU process to reach the value ξ_c [or the value of $\overline{\xi(t)}$ needed for transition as predicted by Ref. [13]] periodically with a period equaling the MFPT to reach that value which is also the $T_{\rm bot}$ for x. Hence we can expect a resonant increase in a statistical quantity, such as $T_{\rm bot}$, but not in a deterministic quantity, such as the position of x (which we take as the output signal in the SR case).

With the confidence given by the above-mentioned physical arguments we searched for a resonance in BSDSCN. But we do not see any peak in the MFPT vs τ curve, and it is monotonic. However, refer to Fig. 2 of Ref. [8] and Fig. 3 of Ref. [12]. The $(S[\tau]-S[0])/S[\infty]$ vs τ curve of Refs. [8] and [12] show a peak. It is tempting to speculate that this peak is due to the resonance between the above-mentioned two factors responsible for escape at intermediate τ. The peak is at $\tau=1$ in Ref. [8] and is between 3 and 7 in Ref. [12].

We supply the following reasons supporting our speculation. We first concentrate on $\hat{\tau}$, the value of τ separating the two adiabatic limits at $x=0$. From the equality $\tau V''(x=0)=1$, we may come to the conclusion that $\hat{\tau}=1$. We now argue that $\hat{\tau}$ is actually D dependent and is equal to 1 only for $D=0$. First note that τ itself is a statistical quantity and the time scale of evolution of $\xi(t)$ itself is time dependent and random and its distribution depends upon D and it equals τ only on the average. Further, only in the limit $D \to 0$, the time scale of evolution of $\xi(t)$ has a narrow distribution around τ and $\hat{\tau}$ becomes 1 and $\hat{\tau}$ always greater than 1 for finite D.

Alternatively, as τ becomes greater than $\hat{\tau}$, the large τ adiabatic limit, and hence the FPT, starts its influence. As τ increases, the region of validity of FPT grows around $x=0$ and reaches $|x|\leq 1/\sqrt{3}$ for $\tau\to\infty$. Also, once x touches the region of validity of FPT, x crosses this region quickly and the transition to the other well is immediate. This argument generalizes the FPT for finite τ. In this context it can also be seen that different theories put forth for extending FPT for finite τ [4,5,13] attempt to find the value of $\overline{\xi(t)}$ needed for x to touch the region of validity of FPT. The tendency of x to cross, rather than to stay in, the region of validity of FPT means x experiences a finite drift toward the other well. As the deterministic drift is small around $x=0$, drift on x has to come from ξ, and therefore the most probable value of ξ at $x=0$ deviates from $\xi=0$ for $\tau\geq\hat{\tau}$, causing a hole in the SPDF of the (x,ξ) for $\tau\geq\hat{\tau}$ [16]. The above arguments bring out the connection between FPT and the hole phenomenon. Similar arguments supporting this connection have been already put forth in Ref. [16(a)]. We further note from Ref. [16(b)] that the minimum value of τ needed for the hole to set in (which is also $\hat{\tau}$) is D dependent and is 1 for $D\to 0$.

Coming back to the value of τ at the peak in $(S[\tau]-S[0])/S[\infty]$, we note that Ref. [8] invokes the limit $D\to 0$. Hence $\hat{\tau}$, the separating point of the two adiabatic regimes (where we expect a resonance to occur), is 1. This supports the speculation that the peak in $(S[\tau]-S[0])/S[\infty]$ is due to a resonance mechanism. It also explains the deviation of the peak value of Ref. [12] from $\tau=1$. The simulation results reported in Ref. [12] were carried out at finite D, and $D\ln(T)$ has been extrapolated to $D=0$ in Fig. 1 of Ref. [12]. Because of this, $S[\tau]$ and $S_{\text{theor}}[\tau]$ of Table III of Ref. [12] differ. This seems to be responsible for the peak in $(S[\tau]-S[0])/S[\infty]$ predicted by the simulation results of Ref. [12] being at a value different from $\tau=1$.

However, it remains to be explained why, if there is a resonance in the escape mechanism at finite τ, it shows up as a peak in the normalized exponent of the MFPT and not in the MFPT itself. Much remains to be understood with regard to the connection between SR and BSDSCN. If indeed a resonance occurs in BSDSCN, it would be very interesting to study and, more than that, challenging to develop the theory behind it.

VII. CONCLUSIONS

Before summarizing the main points of this paper we point out that Eq. (5) underestimates the MFPT for small D, but overestimates the MFPT for large D. Any formula showing such a behavior should be viewed with suspicion because it is bound to estimate the MFPT correctly for some D and any modification in the formula will just shift the point of coincidence. So coincidence does not validate the correctness of the formula. But we point out that the dependency of MFPT on D is rather well understood (see Ref. [12]) and it is the dependency of the MFPT on τ that poses a problem. Since Eq. (5) captures the τ dependency of MFPT very well (though only over a narrow range of D, reasons for which have been given), Eq. (5) escapes from the suspicion raised.

Finally, we summarize the main points of this paper. We have proposed a simple formula for the escape rate in BSDCN valid for finite D and finite τ. The only existing theory for large but finite τ giving satisfactory results is the path-integral method [7,8], but it is restricted to small D. Our approach, which is valid for finite τ and finite D, can be viewed as a complementary approach to the path-integral method. Further, interesting similarities between the stochastic resonance phenomenon and the colored-noise problem are shown.

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